Periodic Orbits on a Surface of Revolution.*

By Daniel Buchanan.

§ 1. *Introduction.* The object of this paper is to determine the periodic orbits described by a particle which moves, subject to gravity, on a smooth surface of revolution, the axis of which is vertical. Let us denote the equation of the surface by

$$x^2 + y^2 - 2pz + 2\epsilon\phi\,(x^2 + y^2, z) = 0, \tag{1}$$

where p is a positive constant, ϵ an arbitrary parameter, and ϕ a power series in $x^2 + y^2$ and z, converging for $x^2 + y^2$ and $|z|$ sufficiently small. There will be no loss of generality if we suppose that the constant term of ϕ is zero, otherwise it could be eliminated by a linear substitution for z, that is, by a translation of the xy-plane along the z-axis. Since the equation (1) vanishes with $x^2 + y^2$ and z, it may be solved, by the theory of implicit functions, for $x^2 + y^2$ as a power series in z converging for $|z|$ sufficiently small. Hence the equation (1) may be expressed as

$$F(x, y, z) = x^2 + y^2 - 2pz + 2\epsilon f(z) = 0, \tag{2}$$

where f is a power series in z converging for $|z|$ sufficiently small.

For $\epsilon = 0$ the equation (2) represents a vertical paraboloid of revolution. The generating parabola has its axis coinciding with the positive z-axis and its semi-latus-rectum equal to p. Periodic orbits are first constructed when the particle moves on this paraboloid. Then the analytic continuation of these orbits is made with respect to ϵ and in this way the orbits for the more general surface (2) are determined. It was for this reason that the parameter ϵ was introduced in (2).

The form of the equation of the surface (2) was suggested by a somewhat similar equation used by Poincaré in his memoir, *Sur les Lignes Géodésiques des Surfaces Convexes: Trans. Am. Math. Soc.,* vol. vi (July, 1905). Geodesic lines play the same rôle in Poincaré's memoir as periodic orbits do in this paper.

It will be readily seen that when $\epsilon = 0$ the problem here considered is somewhat analogous to the well-known problem of the spherical pendulum.†

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† For a complete discussion of the spherical pendulum, including the horizontal as well as the vertical motion, see Moulton's *Periodic Orbits,* chap. III, also *Rendiconti del Circolo Matematico di Palermo,* Vol. XXXII (1911), pp. 338-365.

§ 2. *The Differential Equations.* If the particle is of unit mass and moves without friction, then the differential equations of motion are

$$x'' = X, \quad y'' = Y, \quad z'' = Z - g,$$

where X, Y, and Z are the normal reactions due to the surface, and g is the acceleration due to gravity. Since the surface is assumed to be smooth, the normal reactions at any point are proportional to the direction cosines of the normal at that point, and therefore the differential equations become

$$\left.\begin{aligned} x'' &= X = \lambda F_x = 2\lambda x, \\ y'' &= Y = \lambda F_y = 2\lambda y, \\ z'' &= Z - g = \lambda F_z - g = 2\lambda(-p + \epsilon f_z) - g, \end{aligned}\right\} \tag{3}$$

where λ is a factor of proportionality.

These equations admit the vis viva integral

$$x'^2 + y'^2 + z'^2 = 2g(c - z), \tag{4}$$

where c is the constant of integration.

Since the equation of constraint (2) does not contain t explicitly, the factor λ can be obtained by differentiating $F(x, y, z)$ twice with respect to t and eliminating x'', y'', and z'' from the result by means of the differential equations (3). Then on making use of the relations

$$x^2 + y^2 = 2pz - 2\epsilon f, \quad x'^2 + y'^2 = -z'^2 + 2g(c - z),$$

from (2) and (4), respectively, we find

$$2\lambda = \frac{2g(z - c) - g(p - \epsilon f_z) + z'^2(1 - \epsilon f_{zz})}{p^2 + 2pz - 2\epsilon(f + pf_z) + \epsilon^2 f_z^2}. \tag{5}$$

The part of 2λ which is independent of ϵ is

$$\lambda_0 = \frac{2g(z - c) - gp + z'^2}{p(p + 2z)}. \tag{6}$$

(A). PERIODIC ORBITS ON A PARABOLOID OF REVOLUTION.

§3. *Proof of Existence of Periodic Orbits.* For $\epsilon = 0$ the surface (2) becomes

$$x^2 + y^2 - 2pz = 0, \tag{7}$$

and the differential equations (3) become

$$x'' = \lambda_0 x, \quad y'' = \lambda_0 y, \quad z'' = -p\lambda_0 - g, \tag{8}$$

where λ_0 is defined in (6). The last equation of (8) is independent of the first two and admits the integral

$$z'^2 = \frac{4gz(c-z)+c_1}{p+2z}, \tag{9}$$

where c_1 is the constant of integration.

If the vertical motion is to be periodic, z cannot increase indefinitely and therefore z' must vanish for some value of t, $t=t_0$, say. Suppose $z=z_0$ at $t=t_0$. Then since $z'=0$ at $t=t_0$, it follows that the constant of integration in (9) has the value

$$c_1 = -4gz_0(c-z_0).$$

Hence the integral (9) becomes

$$z' = \pm\sqrt{\frac{4g}{p+2z}(z_0-z)(z_0-c+z)}. \tag{10}$$

It is readily seen from (10) that z' vanishes for $z=z_0$, and also for $z=c-z_0$.

As p is assumed to be positive, no part of the paraboloid (7) will lie below the xy-plane. Then z and z_0 cannot be negative and therefore, for real initial conditions, c cannot be less than z_0. Three cases arise according to the values assigned to c. They are:

Case I. $z_0 \leqq c < 2z_0$.
Case II. $c = 2z_0$.
Case III. $c > 2z_0$.

Case I. Suppose $z_0 \leqq c < 2z_0$ or $c-z_0 < z_0$. Then z' is real so long as z lies in the interval $z_0 \gtreqless z \gtreqless c-z_0$. If the particle is started in the plane $z=z_0$, it cannot remain in this plane for $z=z_0 \neq 0$ does not satisfy the z-equation of (8). Moreover, it cannot move above this plane since $z<z_0$, hence it must move below this plane. Consequently z decreases or z' is negative, and the negative sign must be taken in (10). The particle continues to fall until it reaches the plane $z=c-z_0$ where the velocity again vanishes. As $z=c-z_0$ is not a solution of the z-equation in (8) and as z cannot be less than $c-z_0$ it must increase, or the positive sign must be taken in (10). The particle then rises until the plane $z=z_0$ is reached where the velocity again vanishes and the radical in (10) changes sign. Hence the particle oscillates between the two horizontal planes $z=z_0$ and $z=c-z_0$, the latter plane being the lower plane.

Case II. If $c = 2z_0$ or $c - z_0 = z_0$, then z' would be real only when $z = z_0$ and would therefore be identically zero. In this case the particle would revolve in the horizontal plane $z = z_0$, the centrifugal force and the reaction of the surface being just sufficient to overcome gravity.

Case III. If $c > 2z_0$ or $c - z_0 > z_0$ it can be shown by an argument similar to that used in discussing Case I that the particle oscillates between the same two planes $z = z_0$ and $z = c - z_0$, as in Case I, but the latter plane is the higher plane.

Thus if c is positive and not less than z_0, the vertical motion of the particle is periodic, and therefore the last equation of (8) must admit a periodic solution.

It could also be shown by the usual analytic existence proof that the last equation of (8) admits a periodic solution. It is not necessary, however, to establish the existence of a periodic solution by either method for, by Macmillan's theorem, quoted below, it is shown that if the formal construction of a periodic solution can be made, then this solution will converge under suitable conditions. Macmillan's theorem is as follows: *

If

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n; \mu, t) \quad (i = 1, \cdots, n) \tag{11}$$

is a system of differential equations in which the f_i are expansible as power series in $x_1, \cdots, x_n$, and μ, vanishing for $x_1 = \cdots = x_n = \mu = 0$, with coefficients which are uniform, continuous, and periodic functions of t with the period $2j\pi$: and if the f_i converge for $0 \leqq t \leqq 2j\pi$, when $|x_j| < \rho_j$, $|\mu| < r$, then the solutions $x_i(t)$ are expansible as power series in μ, or any fractional power of μ, which converge for all t in the interval $0 \leqq t \leqq 2j\pi$ provided $|\mu|$ is sufficiently small. If the constants of integration can be determined at each step so as to make the solution formally periodic with the period $2j\pi$, then the solutions so determined will be periodic and converge for all finite values of t provided $|\mu|$ is sufficiently small.

§ 4. *The Vertical Motion in terms of Elliptic Integrals.* The integral (10) may be written

$$\int \frac{\sqrt{p + 2z}\;dz}{\sqrt{(z_0 - z)\,(z - c + z_0)}} = 2\sqrt{g}\int_{t_0}^{t} dt = 2\sqrt{g}(t - t_0). \tag{10'}$$

We desire to express this integral in terms of elliptic integrals. The neces-

* *Trans. Am. Math. Soc.*, Vol. XIII, No. 2, pp. 146-158.

sary substitution is found to be

$$p+2z=\frac{\sqrt{mn}\,(1-\kappa w)}{1+\kappa w},$$

where

$$m=p+2z_0,\ n=p+2c-2z_0,\ \kappa=\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}.$$

Since p, c, and z_0 are all positive, then m is also positive. If $z_0 < p/2+c$ then n is likewise positive, and therefore κ is real and less than 1 numerically. When the above transformation is made in (10′) we obtain

$$-\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\int\frac{(1-\kappa w)\ dw}{(1-\kappa w)\sqrt{(1-w^2)(1-\kappa^2w^2)}}=2\sqrt{g}(t-t_0),$$

which simplifies to

$$\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\left\{I_1-2I_3-\frac{2\kappa}{1-\kappa^2}\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right\}=2\sqrt{g}(t-t_0),$$

where

$$I_1=\int\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the first kind, and

$$I_3=\int\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}},$$

an elliptic integral of the third kind. Thus t can be found for any given value of w or z. For most purposes, however, it is more convenient to have z expressed as a function of t. This will be discussed in the next section.

The half-period, $T/2$, of a complete oscillation can be obtained by integrating (10′) between the limits z_0 and $c-z_0$. The corresponding limits for w are -1 and $+1$. Hence

$$\begin{aligned}2\sqrt{g}(T/2)=\frac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}}\Bigg\{&\int_{-1}^{+1}\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}}\\&-2\int_{-1}^{+1}\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}}\\&-\frac{2\kappa}{1-\kappa^2}\left[\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right]_{-1}^{+1}\Bigg\}.\end{aligned}$$

Since
$$\left[\sqrt{\frac{1-w^2}{1-\kappa^2w^2}}\right]_{-1}^{+1}=0, \qquad \text{then}$$

$$T=\frac{4\sqrt{mn}}{\sqrt{g}(\sqrt{m}+\sqrt{n})}\left\{\int_0^1\frac{dw}{\sqrt{(1-w^2)(1-\kappa^2w^2)}}\right.$$
$$\left.-2\int_0^1\frac{dw}{(1-\kappa^2w^2)\sqrt{(1-w^2)(1-\kappa^2w^2)}}\right\}.$$

§ 5. *Construction of a Periodic Solution for the Vertical Motion.* Let

$$z=v+c/2, \tag{12}$$

where v is the new dependent variable. Then the last equation of (8) becomes

$$v''=-\frac{4gv+v'^2}{k+2v},$$

where $k=p+c$. This equation can be expanded in the form

$$v''=-(1/k)(4gv+v'^2)[1-2v/k+4v^2/k^2 \\ +\cdots(-2v/k)^j+\cdots], \tag{13}$$

which will converge for $|v|<|k/2|$, or $z<p/2+c$. Hence the particle must not rise above the plane $z=p/2+c$, being the plane at the distance c above the horizontal plane passing through the focus of the generating parabola of (7).

Now let

$$v=\gamma w,\ t-t_0=\tfrac{1}{2}\sqrt{k/g\ (1+\delta)}\ \tau, \tag{14}$$

where γ and δ are arbitrary parameters. When (14) is substituted in (13), the factor γ can be cancelled off in the resulting equation, and we obtain

$$\ddot{w}=-[(1+\delta)w+\mu\dot{w}^2][1-2\mu w+\cdots(-2\mu w)^j+\cdots], \tag{15}$$

where $\mu=\gamma/k$, and the dots denote derivation with respect to τ.

The terms of (15) which are independent of μ and δ are $\ddot{w}+w=0$, and the period of the solution is 2π in τ. Now an existence proof, based on Poincaré's extension to Cauchy's theorem, would show that a periodic solution of (15) exists having the form,

$$w=\sum_{j=0}^{\infty}w_j\mu^j,\quad \delta=\sum_{j=1}^{\infty}\delta_j\mu^j, \tag{16}$$

where the w_j are periodic functions of τ with the period 2π, and the δ_j are constants; and that this solution converges for all values of τ in the interval $0\leqq\tau\leqq 2\pi$, provided $|\mu|$ is sufficiently small. Instead of making this rather

simple existence proof, we shall assume the form of solution (16), make the formal construction, and then show that Macmillan's theorem will apply to prove the convergence of the solution.

Since

$$z = v + c/2 = \gamma w + c/2 = k\mu w + c/2, \tag{17}$$

and since $z' = 0$ at $t = t_0$, then at $\tau = 0$ (or $t = t_0$) we may choose,

$$w(0) = a,\ \dot{w}(0) = 0,$$

where a is a real constant. When these initial conditions are imposed on (16), we have

$$w_0(0) = a,\ w_j(0) = 0,\ j = 1, \cdots \infty,\ \dot{w}_j(0) = 0,\ j = 0, 1, \cdots \infty. \tag{18}$$

Let (16) be substituted in (15) and let the resulting equation be denoted by (15′). This equation is an identity in μ and we may equate the coefficients of the same powers of μ, thus obtaining differential equations which define the various w_j. These equations are to be integrated, and the constants of integration and the various δ_j are to be chosen so that the solutions will be periodic and satisfy the initial conditions (18).

Step 0: *Coefficient of* μ^0. The terms of (15′) that are independent of μ give the equation $\ddot{w}_0 + w_0 = 0$, and its solution which satisfies (18) is $w_0 = a \cos \tau$.

Step 1: *Coefficient of* μ. The differential equation obtained from the terms in μ is

$$\ddot{w}_1 + w_1 = -\delta_1 w_0 + 2w_0^2 - \dot{w}_0^2 = -a\delta_1 \cos \tau + (a^2/2)(1 + 3 \cos 2\tau). \tag{19}$$

In order that w_1 shall be periodic the coefficient of $\cos \tau$ in the right member of (19) must be zero, otherwise the solution would contain the non-periodic or Poisson term $\tau \sin \tau$. On putting $\delta_1 = 0$, the general solution of (19) becomes

$$w_1 = A_1 \cos \tau + B_1 \sin \tau + (a^2/2)(1 - \cos 2\tau).$$

From the initial conditions (18) it follows that the constants of integration A_1 and B_1 are both zero, and the desired solution at this step becomes

$$\delta_1 = 0,\ w_1 = (a^2/2)(1 - \cos 2\tau).$$

Step 2: *Coefficient of* μ^2. The differential equation at this step is

$$\begin{aligned}\ddot{w}_2 + w_2 &= -\delta_2 w_0 + 4w_0 w_1 - 4w_0^3 + 2w_0\dot{w}_0^2 - 2\dot{w}_0\dot{w}_1 \\ &= -a(\delta_2 + \tfrac{1}{2}a^2) \cos \tau - \tfrac{7}{2}a^3 \cos 3\tau,\end{aligned}$$

and the desired solution is

$$\delta_2 = -\tfrac{1}{2}a^2, \quad w_2 = \tfrac{7}{16}a^3(-\cos\tau + \cos 3\tau).$$

The remaining steps of the integration are entirely similar to the preceding step and by an induction to the general term it can readily be shown that the process can be carried on indefinitely. It will be observed that δ_j is zero if j is odd, and that each w_j carries the factor a^{j+1} and is a sum of cosines of multiples of τ having the opposite parity of j. The highest multiple of τ in w_j is $(j+1)$.

So far as the computation has been made, the solution of (15) is

$$\left.\begin{aligned} w &= a\cos\tau + (\mu a^2/2)(1-\cos 2\tau) - \frac{7\mu^2a^3}{16}(\cos\tau - \cos 3\tau) + \cdots, \\ \delta &= -\tfrac{1}{2}a^2\mu^2 + \cdots. \end{aligned}\right\} \quad (20)$$

The convergence of this solution will now be considered. In order to put the differential equation (15) in the same form as (11) of Macmillan's theorem, we let

$$w = W_1, \quad dW_1/d\tau = W_2,$$

then (15) becomes

$$\begin{aligned} dW_1/d\tau &= f_1(W_1, W_2; \mu, \tau) = W_2, \\ dW_2/d\tau &= f_2(W_1, W_2; \mu, \tau) = -[(1+\delta)W_1 + W_2^2][1 - 2\mu W_1 + \cdots]. \end{aligned}$$

The f_1 and f_2 are obviously expansible as power series in W_1, W_2, and μ, and vanish for $W_1 = W_2 = \mu = 0$. The coefficients are constants and therefore satisfy the condition that they shall be uniform, continuous, and periodic functions of τ. Further, f_1 and f_2 converge for W_1, W_2, and μ sufficiently small numerically. Hence equations (15) satisfy the conditions of Macmillan's theorem and consequently the solution (20) converges for $|\mu|$ sufficiently small.

When (20) is substituted for w in (17) we obtain

$$z = c/2 + k\left[\mu a\cos\tau + \frac{\mu^2a^2}{2}(1-\cos 2\tau) - \frac{7\mu^3a^3}{16}(\cos\tau - \cos 3\tau) + \cdots\right].$$

As μ and a are both arbitrary and as they occur only in products as indicated, we may suppress either without loss of generality. Let us suppose $a = 1$. Then the periodic solution for the vertical motion of the particle becomes

$$\begin{aligned} z &= c/2 + k[\mu\cos\tau + (\mu^2/2)(1-\cos 2\tau) - (7\mu^3/16)(\cos\tau - \cos 3\tau) \\ &\qquad + \cdots], \\ \delta &= -\tfrac{1}{2}\mu^2 + (\quad)\mu^4 + \cdots. \end{aligned} \quad (21)$$

In the expansion for z, the multiples of τ in the coefficient of μ^j have the same parity as j and therefore

$$z(\pi) = c/2 - k\mu = c/2 - \gamma.$$

Further

$$z(0) = c/2 + k\mu = c/2 + \gamma.$$

Hence the particle oscillates between the planes $z = c/2 + \gamma$ and $z = c/2 - \gamma$. The parameter γ is therefore a scale factor denoting the amplitude of the oscillation on either side of the plane $z = c/2$.

§6. *The Horizontal Motion.* When the solution (21) has been substituted in the first two equations of (8) and the transformation

$$t - t_0 = \tfrac{1}{2}\sqrt{(k/g)(1+\delta)}\,\tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

has been made, we obtain

$$\ddot{x} + k/p[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots]\,x = 0, \tag{22}$$

and the same equation in y, where

$$\theta_1 = -\cos\tau, \ \theta_2 = -\tfrac{1}{8}(1 - \cos 2\tau),$$

and the remaining θ_j are likewise sums of cosines of multiples of τ having the same parity as j, the highest multiple being j.

Equation (22) is similar to the one first discussed by Hill * in 1877 in his celebrated memoir on the motion of the lunar perigee. A very complete list of references to the literature of the differential equations of this type is given by Baker on page 134 of his memoir "On Certain Linear Differential Equations of Astronomical Interest." †

Three cases arise in the solution of (22) depending upon the values of $(k/4p)$.‡ They are

Case I. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ not an integer.
Case II. $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and $\sqrt{k/p}$ an integer.
Case III. $\tfrac{1}{2}\sqrt{k/p} = 0$.

Since $k = p + c$ and since p and c are both positive in the physical problem under consideration, it follows that $\tfrac{1}{2}\sqrt{k/p} \neq 0$ and Case III need not be considered.

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* *The Collected Works of G. W. Hill,* Vol. I, pp. 243-270; *Acta Mathematica,* Vol. VIII, pp. 1-36.

† *Phil. Trans. of the Royal Society of London,* Series A, Vol. 216, pp. 129-186.

‡ Compare also Moulton's *Periodic Orbits,* Chap. III, § 52.

Case I.—This may be regarded as the general case.

The form of the solution of (22), first given by Floquet,* is

$$x = e^{i\alpha\tau}u, \tag{23}$$

where α and u are power series in μ, the former having constant coefficients, the latter periodic coefficients with the period 2π in τ. When (23) is substituted in (22) we obtain

$$\ddot{u} + 2i\alpha\dot{u} - \alpha^2 u + k/p[\tfrac{1}{4} + \theta_1\mu + \theta_2\mu^2 + \cdots + \theta_j\mu^j + \cdots]u = 0. \tag{24}$$

Now let

$$\begin{aligned} \alpha &= \tfrac{1}{2}\sqrt{k/p} + \alpha_1\mu + \alpha_2\mu^2 + \cdots, \\ u &= u_0 + u_1\mu + u_2\mu^2 + \cdots, \end{aligned} \tag{25}$$

be substituted in (24). We obtain a differential equation (24′), which is an identity in μ. Since the right number is zero, then the coefficient of each power of μ must also be zero. On solving the various differential equations thus obtained, we determine the u_j and α_j, the u_j as periodic functions of τ having the period 2π, and the α_j as constants so determined that the u_j shall be periodic.

Since the solution (23) is later multiplied by an arbitrary constant, see equations (39), we may choose $u(0) = 1$, from which it follows that

$$u_0(0) = 1, \; u_j(0) = 0, \; j = 1, \cdots \infty. \tag{26}$$

Step 0: *Coefficient* of μ^0. From the terms of (24′) that are independent of μ, we have the differential equation

$$\ddot{u}_0 + i\sqrt{k/p}\;\dot{u}_0 = 0,$$

and its solution is

$$u_0 = a_0 + b_0 e^{-i\sqrt{k/p}\;\tau}, \tag{27}$$

where a_0 and b_0 are the constants of integration. Since $\sqrt{k/p}$ is not an integer in this case, the term $e^{-i\sqrt{k/p}\tau}$ does not have the period 2π and we put $b_0 = 0$. From (26) it follows that $a_0 = 1$ and the desired solution at this step becomes $u_0 = 1$.

Step 1: *Coefficient of* μ. On equating to zero the terms in μ in (24′) we obtain

$$\ddot{u}_1 + i\sqrt{k/p}\;\dot{u}_1 = \sqrt{k/p}\;\alpha_1 + k/p\;\cos\tau. \tag{28}$$

* *Annales de l'École Normale Supérieure*, 1883-4.

The complementary function is

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau},$$

a_1 and b_1 being arbitrary constants.

It is well known in the theory of differential equations that the presence of any term in the right member which has *exactly* the same period as any term of the complementary function will yield Poisson terms in the particular integral, that is, terms containing τ outside of trigonometric or exponential symbols. Since the constant part of the right member of (28) has *exactly* the same period as the constant in the complementary function, then non-periodic terms will arise in the particular integral unless $\alpha_1 = 0$. On putting $\alpha_1 = 0$ the complete solution of (28) becomes

$$u_1 = a_1 + b_1 e^{-i\sqrt{k/p}\,\tau} + \frac{k}{k-p}\left[\cos\tau - i\sqrt{k/p}\sin\tau\right]. \tag{29}$$

From the periodicity and initial conditions it follows that

$$b_1 = 0,\ a_1 = \frac{k}{p-k}$$

and the desired solution at this step becomes

$$u_1 = \frac{k}{p-k}\left[1 - \cos\tau + i\sqrt{k/p}\sin\tau\right],\ \alpha_1 = 0. \tag{30}$$

Step 2: Coefficient of μ^2. The differential equation at this step is

$$\begin{aligned}\ddot{u}_2 + i\sqrt{k/p}\,\dot{u}_2 &= \alpha_2\sqrt{k/p} + \frac{k}{8p} - \frac{k^2}{2p(p-k)}\\ &+ \frac{k^2}{p(p-k)}\left(\cos\tau - \tfrac{1}{2}\cos 2\tau + \tfrac{1}{2}\,i\sqrt{k/p}\sin 2\tau\right) - (k/p)\cos 2\tau\ ,\end{aligned} \tag{31}$$

and the solution which satisfies the periodicity and initial conditions is

$$\left.\begin{aligned}u_2 &= \frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}\left(\cos\tau - i\sqrt{k/p}\sin\tau\right)\\ &+ \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau + i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau,\\ \alpha_2 &= \tfrac{1}{8}\sqrt{k/p}\left(\frac{5k-p}{p-k}\right).\end{aligned}\right\} \tag{32}$$

So far as the computation has been carried out it is observed that the

a_j are real constants and that the u_j consist of sums of cosines and i times sines of multiples of τ, the highest multiple being j. It will now be shown by induction that these properties hold in general.

Let us suppose that $a_1, \cdots, a_{n-1}, u_0, u_1, \cdots, u_{n-1}$ have been computed and that the various a_j are real and that the u_j, $j = 1, \cdots, n-1$, have the form

$$u_j = \sum_{l=0}^{j} (A_j^{(l)} \cos l\tau + iB_j^{(l)} \sin l\tau), \tag{33}$$

where the $A_j^{(l)}$ and $B_j^{(l)}$ are real constants. We wish to show that a_n can be determined as a real constant and that u_n has the same form as (33) when $j = n$.

Step n: Coefficient of μ^n. The differential equation at this step is

$$\ddot{u}_n + i\sqrt{k/p}\,\dot{u}_n = a_n\sqrt{k/p} + U_n(a_1, \cdots, a_{n-1};\, u_1, \cdots, u_{n-1}). \tag{34}$$

The only undetermined constant which enters (34) is a_n and it is written explicitly in so far as it occurs. The function U_n is linear in $u_1, \cdots, u_{n-1}$. The terms of U_n which arise from $2ia\dot{u}$ in (24) have the form

$$2ia_l u_{n-l}, \quad l = 1, \cdots, n-1,$$

and are cosines and i times sines of multiples of τ, the highest multiple being $n-1$. The terms which arise from a^2u in (24) have the form

$$-(\ \ )a_l\, a_m\, u_{n-(l+m)}, \quad l, m = 0, \cdots, n-1, \quad l+m \leqq n-1;$$

where ( ) denotes 1 if $l = m$ and 2 if $l \neq m$. These terms have the same form as the preceding terms. When $l = m = 0$, the term $-a_0^2 u_n$ will cancel off with the term $(k/4p)u_n$ arising from the last factor of (24) and therefore the highest multiple of τ in the sines and cosines is $n-1$. The last factor of (24) gives, in addition to the term $(k/4p)u_n$ just considered, terms of the type

$$(k/p)\,\theta_l u_{n-l}, \quad l = 1, \cdots, n-1.$$

These terms are also of the same form as (33) but the highest multiple of τ which they yield is n. Therefore

$$U_n = \sum_{l=0}^{n} [a_l^{(n)} \cos l\tau + ib_l^{(n)} \sin l\tau], \tag{35}$$

where $a_l^{(n)}$ and $b_l^{(n)}$ are real constants.

Let us now determine the periodic solution of (34). In order that the

solution shall be periodic, the right member of (34) must contain no constant terms. Hence

$$a_n = -\sqrt{p/k}\, a_0^{(n)},$$

a real constant. The complete solution of (34) then becomes

$$u_n = a_n + b_n e^{-i\sqrt{k/p}\,\tau} + \bar{U}_n,$$

where a_n and b_n are the constants of integration, and the particular integral $\bar{U}_n$ has the same form as (35), except that it has no constant term. From the periodicity and the initial conditions we have

$$b_n = 0,\ a_n = -\bar{U}_n(0),$$

and the desired solution for u_n is the same as when $j = n$. Hence the properties of a_j and u_j already stated hold in general.

A second solution of (22) could be constructed in an entirely similar way but it is not necessary to make the construction as this solution can be obtained directly from the former solution.

Let the solution of (24) already obtained be denoted by $u(\tau, +i)$. Then one solution of (22) is

$$x = e^{+ia\tau}\, u(\tau, +i). \tag{36}$$

Since the differential equation (22) is independent of i, a change in the sign of i in (36) will still give a solution, viz.,

$$x = e^{-ia\tau}\, u(\tau, -i). \tag{37}$$

Thus a second solution can be obtained by changing the sign of i in the first solution.

The determinant formed by the two solutions (36) and (37) together with their derivatives with respect to τ is a constant,* and its value can be computed most readily when $\tau = 0$. This determinant is

$$D = -2[ia + \dot{u}(0, +i)] = -i\sqrt{k/p} + \text{terms in } \mu, \tag{38}$$

which is different from zero for $\mu = 0$ and therefore remains different from zero for $|\mu|$ sufficiently small. The two solutions (36) and (37) therefore constitute a fundamental set and the most general solution of (22), as well as of the similar equation in y, is

* Moulton's *Periodic Orbits*, § 18.

$$\left.\begin{aligned}
&x = A_1 e^{ia\tau} u^{(1)} + A_2 e^{-ia\tau} u^{(2)},\\
&y = B_1 e^{ia\tau} u^{(1)} + B_2 e^{-ia\tau} u^{(2)},\\
&a = \tfrac{1}{2}\sqrt{k/p}\,[1 + \frac{5k-p}{4(p-k)}\mu^2 + \cdots],\\
&u^{(1)} = u(\tau, +i) = 1 + \frac{k}{k-p}[1 - \cos\tau + i\sqrt{k/p}\sin\tau]\mu\\
&\quad + \Big[\frac{7pk^3 - 22p^2k^2 + 4p^3k - k^4}{4p(p-k)^2(k-4p)} - \frac{k^2}{(p-k)^2}(\cos\tau - i\sqrt{k/p}\sin\tau)\\
&\quad + \frac{2pk^2 - 4p^2k - k^3}{4p(p-k)(k-4p)}\cos 2\tau\\
&\quad + i\sqrt{k/p}\,\frac{k(k+2p)}{4(p-k)(k-4p)}\sin 2\tau)\Big]\mu^2 + \cdots,\\
&u^{(2)}(+i) = u^{(1)}(-i),
\end{aligned}\right\} \tag{39}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

By using Macmillan's theorem, quoted in § 3, it can readily be shown that the above solutions converge for all τ in the interval $0 \leqq \tau \leqq 2\pi$ provided $|\mu|$ is sufficiently small.

Case II. Suppose $\sqrt{k/p} = \nu$, an integer. Since $k = p + c$ and p and c are both positive, then $\nu > 1$. Two sub-cases arise depending upon the value of ν. They are:

Sub-case I. $\nu = 2$,
Sub-case II. $\nu \neq 2$.

Sub-case I. The construction proceeds as in Case I until equation (27) is reached, and this becomes

$$u_0 = a_0 + b_0 e^{-2i\tau}, \tag{40}$$

where both terms have the period 2π. From the initial condition $u_0(0) = 1$ it follows that $b_0 = 1 - a_0$, and the solution for u_0 becomes

$$u_0 = a_0 + (1 - a_0)e^{-2i\tau}, \tag{41}$$

with the constant a_0 remaining arbitrary at this step.

Step 1: *Coefficient of* μ. When we equate to zero the coefficient of μ in (24″), that is in equation (24) after (25) has been substituted and $\sqrt{k/p}$ replaced by 2, we obtain

$$\ddot{u}_1 + 2i\dot{u} = 2a_1a_0 - 2a_1(1 - a_0)e^{-2i\tau} + 2a_0e^{i\tau} + 2e^{-i\tau} + 2(1 - a_0)e^{-3i\tau}. \tag{42}$$

The complementary function of this equation is

$$u_1 = a_1 + b_1 e^{-2i\tau}.$$

Since terms of *exactly* the same period as those of the complementary function occur in the right member of (42), viz., constants and terms in $e^{-2i\tau}$, the particular integral will contain non-periodic terms unless the constants and the coefficient of $e^{-2i\tau}$ are put equal to zero. Since we seek a periodic solution we therefore put

$$2a_1 a_0 = 0, \; 2a_1(1 - a_0) = 0. \tag{43}$$

These equations are satisfied by $a_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 or this case would be the same as Case I. The complete solution of (42) then becomes

$$u_1 = a_1 + b_1 e^{-2i\tau} - \tfrac{2}{3} a_0 e^{i\tau} + 2e^{-i\tau} - \tfrac{2}{3}(1 - a_0) e^{-3i\tau}. \tag{44}$$

Since $u_1(0) = 0$, then

$$b_1 = -(\tfrac{4}{3} + a_0).$$

At this step a_0 and a_1 still remain undetermined.

Step 2: Coefficient of μ^2. The terms of (24″) which contain the factor μ^2 give the equation

$$\begin{aligned}\ddot{u}_2 + 2i\dot{u}_2 = {} & [2a_2 a_0 + \tfrac{7}{6} a_0 + 2] + [-2a_2(1 - a_0) - \tfrac{7}{6} a_0 + \tfrac{19}{6}] e^{-2i\tau} \\ & + 2a_1 e^{i\tau} - \tfrac{10}{3} a_0 e^{2i\tau} - \tfrac{8}{3} e^{-i\tau} \\ & - 2(a_1 + \tfrac{4}{3}) e^{-3i\tau} - \tfrac{10}{3}(1 - a_0) e^{-4i\tau}.\end{aligned}$$

On equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member, as at the preceding step, we have

$$\left.\begin{aligned} & 2a_2 a_0 - \tfrac{7}{6} a_0 + 2 = 0, \\ & -2a_2(1 - a_0) - \tfrac{7}{6} a_0 + \tfrac{19}{6} = 0. \end{aligned}\right\} \tag{45}$$

These equations are satisfied by

$$a_2 = \pm \tfrac{1}{12}\sqrt{217}, \quad a_0 = \tfrac{1}{2} \mp \tfrac{1}{14}\sqrt{217}, \tag{46}$$

where the upper signs are to be taken together, also the lower. When equations (46) are satisfied, the general solution for u_2 will be periodic, being

$$\begin{aligned}u_2 = {} & a_2 + b_2 e^{-2i\tau} - \tfrac{2}{3} a_1 e^{i\tau} + \tfrac{5}{12} a_0 e^{2i\tau} - \tfrac{8}{3} a_0 e^{-i\tau} \\ & + \tfrac{2}{3}(a_1 + \tfrac{4}{3}) e^{-3i\tau} + \tfrac{5}{12}(1 - a_0) e^{-4i\tau}.\end{aligned}$$

Since $u_2(0) = 0$ then

$$b_2 = -a_2 + 8/3 a_0 - 47/36.$$

The undetermined constants at this step are a_2 and α_1.

Step 3: *Coefficient of* μ^3. The differential equation at this step is

$$\ddot{u}_3 + 2i\dot{u}_3 = U_3 = -2i\alpha_3\dot{u}_0 - 2i\alpha_2\dot{u}_1 + 2a_3u_0 + 2a_2u_1 \\ - u_0\theta_3 + u_1(\tfrac{1}{2} - 4\cos 2\tau) + 4u_2\cos\tau.$$

If we consider only the part of U_3 which involves the arbitrary constants that are determined at this step we have

$$\begin{aligned} U_3 = {} & 2a_3a_0 + \alpha_1(2a_2 + 7/6) + d_1 \\ & + [-2(1-a_0)\alpha_3 + \alpha_1(2a_2 - 7/6) + d_2e^{-2i\tau}] \\ & + \text{terms in } e^{\pm i\tau},\ e^{2i\tau},\ e^{\pm 3i\tau},\ e^{\pm 5i\tau},\ e^{-4i\tau}, \end{aligned} \tag{47}$$

where d_1 and d_2 are known constants. To make u_3 periodic we equate to zero the constants and the coefficient of $e^{-2i\tau}$ in (47) giving

$$\left.\begin{aligned} 2a_3a_0 + \alpha_1(2a_2 + 7/6) + d_1 = 0, \\ -2(1-a_0)\alpha_3 + \alpha_1(2a_2 - 7/6) + d_2 = 0. \end{aligned}\right\} \tag{48}$$

The determinant of the coefficients of α_3 and α_1 in (48) is

$$D_1 = \pm\, 2/3\sqrt{217}, \tag{49}$$

where the upper sign here is to be taken with the upper signs in (46), similarly for the lower signs. Since this determinant D_1 is different from zero, equations (48) can be solved for α_3 and α_1, and the solutions are unique for either set of values in (46). When (48) is satisfied, the solution for u_3 will therefore be periodic. Two constants of integration, a_3 and b_3, say, will arise from the complementary function. One constant, b_3, say, will be determined from the initial condition $u_3(0) = 0$, while α_2 and a_3 remain undetermined at this step.

The succeeding steps of the integration are entirely similar to step 3 just considered. So far as the constants of integration are concerned one of them, b_j say, is determined by the initial condition $u_j(0) = 0$ at the step where it arises. The other constant a_j of the step j, is not determined until the step $j+2$ is reached, where, on equating to zero the constants and the coefficient of $e^{-2i\tau}$ in the right member of the differential equation in u_{j+2}, two linear equations in α_{j+2} and a_j are obtained and the determinant of their coefficients is the same as D_1 in (49). Hence α_{j+2} and a_j can be uniquely determined provided one set of values is taken in (46).

It is thus evident that two solutions of (24″) can be obtained in the foregoing construction, according as the upper or lower signs are taken in (46). The corresponding solutions of (22) are

$$x = x_1 = e^{i(1+1/12\sqrt{217}\,\mu^2+\cdots)\tau}\,[(1/2 - 1/14\sqrt{217}) + (1/2 + 1/14\sqrt{217})e^{-2i\tau} + (\ )\mu + \cdots],$$

and

$$x = x_2 = e^{i(1-1/12\sqrt{217}\,\mu^2+\cdots)\tau}\,[(1/2 + 1/14\sqrt{217}) + (1/2 - 1/14\sqrt{217})e^{-2i\tau} + (\ )\mu + \cdots].$$

If we proceed as in Case I to obtain two solutions by changing the sign of i in the preceding solutions it would appear that two additional solutions could be obtained, but this is impossible since the differential equation (22) is only of the second order and admits of but two solutions. This apparent difficulty is overcome if the factor $e^{i\tau}$ is multiplied into the part of the solutions contained in the square brackets [ ]. Thus

$$\left.\begin{aligned} x_1 &= e^{i(1/12\sqrt{217}\,\mu^2+\cdots)\tau}\,[(1/2 - 1/14\sqrt{217})e^{i\tau} + \\ &\quad + (1/2 + 1/14\sqrt{217})e^{-i\tau} + (\ )\mu + \cdots], \\ x_2 &= e^{-i(1/12\sqrt{217}\,\mu^2+\cdots)\tau}\,[(1/2 + 1/14\sqrt{217})e^{i\tau} + \\ &\quad + (1/2 - 1/14\sqrt{217})e^{-i\tau} + (\ )\mu + \cdots]. \end{aligned}\right\} \tag{50}$$

It is obvious, therefore, that the two solutions x_1 and x_2 differ only in the sign of i or of $\sqrt{217}$.

If the exponentials in the square brackets of (50) are expressed in trigonometric form, these solutions become

$$\left.\begin{aligned} & x_1 = e^{ia\tau}u^{(1)},\ x_2 = e^{-ia\tau}u^{(2)}, \\ & a = 1/12\sqrt{217}\,\mu^2 + \cdots, \\ & u^{(1)} = \cos\tau - i\,1/7\sqrt{217}\sin\tau + (\ )\mu + \cdots, \\ & u^{(2)}(+i) = u^{(1)}(-i). \end{aligned}\right\} \tag{51}$$

The terms represented by $u^{(1)}$ and $u^{(2)}$ differ not only in i but also in the sign of $\sqrt{217}$. They are power series in μ with sums of cosines and $i\sqrt{217}$ times sines of multiples of τ, the highest multiple in the coefficient of μ^j being $j+1$.

The determinant of the two solutions in (51) together with their derivatives is a constant, as in Case I, and its value at $\tau = 0$ is

$$D_2 = -2[ia + \dot{u}^{(1)}(0)] = i\,2/7\sqrt{217} + (\ )\mu + \cdots,$$

which is different from zero for $\mu = 0$, and therefore remains different from zero for $|\mu|$ sufficiently small. Hence the two solutions (51) constitute a fundamental set, and the most general solutions of (22) and the similar equation in y are

$$x = A_1 e^{i\alpha\tau} u^{(1)} + A_2 e^{-i\alpha\tau} u^{(2)}, \quad y = B_1 e^{i\alpha\tau} u^{(1)} + B_2 e^{-i\alpha\tau} u^{(2)}, \tag{52}$$

where α, $u^{(1)}$, and $u^{(2)}$ are defined in (51), and A_1, A_2, B_1, B_2 are the constants of integration.

Sub-case II. When ν is an integer different from 2, the construction is the same as in the preceding sub-case until equations (43) are reached. The equations analogous to (43) are

$$\nu\alpha_1 a_0 = 0, \qquad \nu\alpha_1(1 - a_0) = 0. \tag{53}$$

These are satisfied by $\alpha_1 = 0$, a_0 arbitrary, but a_0 must be different from 1 as in the preceding sub-case.

The equations which correspond to (45) are

$$\left.\begin{aligned} a_0 \left\{ \nu\alpha_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0, \\ (1 - a_0) \left\{ -\nu\alpha_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} &= 0. \end{aligned}\right\} \tag{54}$$

Since $a_0 \neq 1$ in Case II, then the solutions of (54) are

$$a_0 = 0, \ \alpha_2 = \frac{\nu(5\nu^2 - 1)}{8(\nu^2 - 1)}. \tag{55}$$

The equations analogous to (48) are

$$a_1 \left\{ \nu\alpha_2 + \frac{\nu^2(5\nu^2 - 1)}{8(\nu^2 - 1)} \right\} = 0, \qquad -\nu\alpha_3 + \nu^2\theta_3^{(0)} = 0, \tag{56}$$

where $\theta_3^{(0)}$ is the constant part of θ_3 in equation (22). The solutions of these equations are

$$a_1 = 0, \ \alpha_3 = \nu\theta_3^{(0)}.$$

The constants a_2 and α_4 are determined in the same way at the next step as a_1 and α_3 were found in the preceding step. They have the same coefficient in the equations analogous to (56) as a_1 and α_3 have in (56), and similarly for the succeeding steps. It will be found that

$$a_j = 0, \ j = 0, 1, \cdots, \nu - 3,$$

and that the remaining a_j are, in general, different from zero.

So far as the computation has been made we find

$$\left.\begin{aligned}
&u_0 = e^{-i\nu\tau},\\
&u_1 = \frac{\nu^2}{\nu^2-1} e^{-i\nu\tau} + \nu^2/2 \left[\frac{e^{-i(\nu-1)\tau}}{\nu-1} - \frac{e^{-i(\nu+1)\tau}}{\nu+1}\right],\\
&u_2 = \left[\frac{\nu^4}{(\nu^2-1)^2} + \frac{\nu^2}{8(\nu^2-4)}\right] e^{-i\nu\tau} - \frac{\nu^4}{2(\nu^2-1)}\left[\frac{e^{-i(\nu-1)\tau}}{\nu-1}\right.\\
&\quad\left. - \frac{e^{-i(\nu+1)\tau}}{\nu+1}\right] - \nu^2/32 \left[\frac{e^{-i(\nu-2)\tau}}{\nu-2} - \frac{e^{-i(\nu+2)\tau}}{\nu+2}\right],\\
&a_1 = 0,\ a_2 = \frac{\nu(5\nu^2-1)}{8(\nu^2-1}.
\end{aligned}\right\}$$

When these terms are substituted in (25) and the result in (23) we obtain one solution of (22). Let it be denoted by

$$x = e^{ia\tau}u^{(1)}. \tag{58}$$

Another solution can be obtained by changing the sign of i in (58), thus

$$x = e^{-ia\tau}u^{(2)}, \tag{59}$$

where $$u^{(2)}(+i) = u^{(1)}(-i).$$

The determinant of these two solutions and their first derivatives is

$$-2[ia + \dot{u}^{(1)}(0)] = i\nu + (\quad)\mu + \cdots,$$

which is different from zero for $|\mu|$ sufficiently small. The solutions (58) and (59) therefore constitute a fundamental set, and the general solutions of (22) and the corresponding equation in y are of the same form as (39) or (52), viz.,

$$x = A_1 e^{ia\tau}u^{(1)} + A_2 e^{-ia\tau}u^{(2)},\ y = B_1 e^{ia\tau}u^{(1)} + B_2 e^{-ia\tau}u^{(2)}, \tag{60}$$

where a and $u^{(1)}$ have the values in (58). If the exponentials in (58) are expressed in trigonometric form then

$$u^{(1)} = \sum_{j=0}^{\infty}\sum_{l=0}^{j}[A^{(j)} + B^{(j)}_{\pm l}\cos(\nu \pm l)\tau + iC^{(j)}_{\pm l}\sin(\nu \pm l)\tau]\mu^j, \tag{61}$$

where $A^{(j)}$, $B^{(j)}_{\pm l}$, and $C^{(j)}_{\pm l}$ are real constants. In particular

$$A^{(j)} = 0,\ j = 0, 1, 2,\ B^{(j)}_{\pm l} = -C^{(j)}_{\pm l},\ j = 0, 1, 2,\ l = 0, 1, 2,$$

$$B^{(0)}_{0} = 1,\ B^{(1)}_{0} = -\frac{\nu^2}{\nu^2-1},$$

$$B^{(1)}_{-1} = \frac{\nu^2}{2(\nu-1)},\ B^{(1)}_{-1} = -\frac{\nu^2}{2(\nu+1)},$$

$$B^{(2)}_{0}=\frac{\nu^4}{(\nu^2-1)^2}+\frac{\nu^4}{8(\nu^2-4)},\ B^{(2)}_{-1}=-\frac{\nu^4}{2(\nu-1)(\nu^2-1)},$$

$$B^{(2)}_{+1}=-\frac{\nu^4}{2(\nu+1)(\nu^2-1)},\ B^{(2)}_{-2}=-\frac{\nu^2}{32(\nu-2)},$$

$$B^{(2)}_{+2}=\frac{\nu^2}{32(\nu+2)}.$$

From (57) it follows that

$$A^{(j)}=0,\ j=0,1,\cdots,\nu-3.$$

The remaining $A^{(j)}$ are different from zero, in general.

If the factors $e^{i(\nu/2)\tau}$ and $e^{-i(\nu/2)\tau}$ are taken with $u^{(1)}$ and $u^{(2)}$, respectively, in (60), then $u^{(1)}$ and $u^{(2)}$ will have cosines and sines of $(\nu/2\pm l)\tau$ instead of $(\nu\pm l)\tau$ as in (61).

§ 7. *The Arbitrary Constants.* Each of the solutions for x, y, and z contains two arbitrary constants, and the values of these constants for the physical problem will now be considered.

In this discussion no discrimination will be made between the two cases of the preceding section. It was for this reason that the same notation was chosen for the solutions (39), (52), and (60).

Besides satisfying the differential equations (8), the solutions for x, y, and z must also satisfy (7), viz.,

$$x^2+y^2-2pz=0, \tag{7}$$

and their derivatives with respect to t must satisfy the *vis viva* integral (4). When the transformation

$$t-t_0=\tfrac{1}{2}\sqrt{k/g(1+\delta)}\,\tau,\ \delta=-\tfrac{1}{2}\mu^2+\cdots$$

is made in (4), this integral becomes

$$\dot{x}^2+\dot{y}^2+\dot{z}^2=(k/2)(1+\delta)(c-z). \tag{62}$$

Since any horizontal section of the surface (7) is a circle with centre on the z-axis, the xy-axes may be rotated about the z-axis without changing the form of (7), and we may therefore suppose that the particle lies in the xz-plane at the initial time, or $y=0$ at $\tau=0$. Since $u^{(1)}(0)=u^{(2)}(0)=1$, then it follows that $B_1+B_2=0$.

As $\dot{z}(0)=0$, the particle must be initially projected in a horizontal plane. Since any horizontal section of the surface of constraint is a circle with centre on the z-axis and since $u(0)=0$, then it follows that $\dot{x}(0)=0$. When this condition is imposed on the solution for x, we have

$$[ia + \dot{u}^{(1)}(0)]\ [A_1 - A_2] = 0.$$

The factor $[ia + \dot{u}^{(1)}(0)]$ is different from zero for $|\mu|$ sufficiently small and therefore

$$A_1 = A_2.$$

Hence the solutions for x and y become

$$x = A[e^{ia\tau}u^{(1)} + e^{-ia\tau}u^{(2)}],\ y = B[e^{ia\tau}u^{(1)} - e^{-ia\tau}u^{(2)}], \tag{63}$$

where the now unnecessary subscripts on the constants A and B have been dropped.

On putting $\tau = 0$ in (63) and (21) we get

$$\left.\begin{aligned} &x(0) = 2A, && \dot{x}(0) = 0, \\ &y(0) = 0, && \dot{y}(0) = 2B[ia + \dot{u}^{(1)}(0)], \\ &z(0) = c/2 + k\mu, && \dot{z}(0) = 0. \end{aligned}\right\} \tag{64}$$

When these values are substituted in (7) and (62) we find

$$\begin{aligned} A &= \pm \tfrac{1}{2}\sqrt{p(c + 2k\mu)} = \pm \tfrac{1}{2}\sqrt{p(c + 2\gamma)}, \\ B &= \pm \frac{\sqrt{k(1+\delta)(c - 2\gamma)}}{4[ia + \dot{u}^{(1)}(0)]} = \mp i\frac{\sqrt{k(1+\delta)(c - 2\gamma)}}{4[a - i\dot{u}^{(1)}(0)]}. \end{aligned}$$

Thus A is real and B is purely imaginary. Hence the solutions of equations (8), in terms of τ, are

$$\left.\begin{aligned} x &= \pm \tfrac{1}{2}\sqrt{p(c + 2\gamma)}\ [e^{ia\tau}u^{(1)} + e^{-ia\tau}u^{(2)}], \\ y &= \mp \frac{i\sqrt{k(1+\delta)(c - 2\gamma)}}{4[a - i\dot{u}^{(1)}(0)]}\ [e^{ia\tau}u^{(1)} - e^{-ia\tau}u^{(2)}], \\ z &= c/2 + k[\mu \cos\tau + \tfrac{1}{2}\mu^2(1 - \cos 2\tau) \\ &\qquad\qquad - \tfrac{7}{16}\mu^3(\cos\tau - \cos 3\tau) + \cdots], \\ \delta &= -\tfrac{1}{2}\mu^2 + \cdots, \\ \mu &= \gamma/k, \qquad\qquad k = p + c. \end{aligned}\right\} \tag{65}$$

The terms a, $u^{(1)}$, and $u^{(2)}$ are defined in (39), (52), and (60) according to the value of $\sqrt{k/p}$. The double signs in (65) depend upon the octants of space into which the particle is initially projected.

If the exponentials in (65) are expressed in trigonometric form, the solutions of (22) for x and y become

$$\begin{aligned} x &= \sqrt{p(c + 2\gamma)}\ [\{\ \}_1 \cos a\tau - \{\ \}_2 \sin a\tau], \\ y &= \frac{\sqrt{k(1+\delta)(c - 2\gamma)}}{2[a - i\dot{u}^{(1)}(0)]}\ [\{\ \}_1 \sin a\tau + \{\ \}_2 \cos a\tau], \end{aligned}$$

where $\{\ \}_1$ and $\{\ \}_2$ are power series in μ which have different values for the different cases of § 6. For Case I,

$$\{\ \}_1 = 1 + \sum_{j=1}^{\infty} \sum_{l=0}^{j} [a_{jl} \cos l\tau]\mu^j,$$

$$\{\ \}_2 = \sum_{j=1}^{\infty} \sum_{l=1}^{j} [b_{jl} \sin l\tau]\mu^j,$$

where a_{jl} and b_{jl} are the coefficients of cos $l\tau$ and sin $l\tau$, respectively, in u_j, equations (30), (32), and (33).

For Case II,

$$\{\ \}_1 = \sum_{j=0}^{\infty} \sum_{l=0}^{j} [A^{(j)} + B_{\pm l}^{(j)} \cos(\nu \pm l)\tau]\mu^j,$$

$$\{\ \}_2 = \sum_{j=1}^{\infty} \sum_{l=0}^{j} [C_{\pm l}^{(j)} \sin(\nu \pm l)\tau]\mu^j.$$

The constants $A^{(j)}$, $B_{\pm l}^{(j)}$, and $C_{\pm l}^{(j)}$ have the same values as in (61) for sub-case II, and for sub-case I they are, in so far as the computation has been carried out,

$$A^{(0)} = 0,\ B_0^{(0)} = 1,\ C_0^{(0)} = -\tfrac{1}{7}\sqrt{217}.$$

There still remain in the solutions (65) two constants which have not been determined. They are c and γ $(= k\mu)$. Since c is the constant of integration arising in the *vis viva* integral, its value depends upon the initial velocity of the particle. As $x' = z' = 0$ at $t = t_0$, the initial velocity may be denoted by y_0'. Then from the *vis viva* integral (4) it follows that

$$y_0'^2 = 2g(c - z_0) = g(c - 2\gamma), \text{ or } c - 2\gamma = y_0'^2/g.$$

Since, further, $c + 2\gamma = 2z_0$, then

$$c = z_0 + y_0'^2/2g,\ \ \gamma = \tfrac{1}{2}(z_0 - y_0'^2/2g).$$

Thus the constants c and γ are functions of the initial velocity and the initial height.

Let us now return to the three cases of § 3. Since $z_0 = c/2 + \gamma$ these cases become:

Case I. $0 < \gamma \leqq c/2$,
Case II. $\gamma = 0$,
Case III. $\gamma < 0,\ |\gamma| \leqq c/2$.

In Case I and Case III the particle oscillates between the horizontal planes $z = c/2 + \gamma$ and $z = c/2 - \gamma$. The two orbits are geometrically the same but the motion in the one orbit is half a period ahead of the motion in the

other. If $\gamma = c/2$, then the initial velocity is zero and as there is no lateral projection the particle will move in the vertical parabola $x^2 = 2pz$. If $\gamma = -c/2$, then $z_0 = 0$, and the particle is projected from the lowest point with the initial velocity $y'_0 = \sqrt{2gc}$. It will therefore move in the vertical parabola $y^2 = 2pz$ which is dynamically the same orbit as $x^2 = 2pz$. The orbits when $\gamma = \pm c/2$ correspond therefore to the simple pendulum.

In Case II when $\gamma = 0$, then $z = c/2$, a constant, and the differential equations of motion become

$$x'' + (g/p)x = 0, \; y'' + (g/p)y = 0, \; z'' = 0.$$

Their periodic solutions which satisfy the initial conditions are

$$x = \sqrt{pc}\,\cos\sqrt{g/p}\,t, \quad y = \sqrt{pc}\,\sin\sqrt{g/p}\,t, \quad z = c/2.$$

In this case the particle moves in a circle the plane of which is parallel to the xy-plane and at a distance $c/2$ above it.

In the first paragraph of § 5 it is stated that the inequality $z < p/2 + c$ must hold in order that the expression in (13) will converge. Now the maximum value of z is

$$z = c/2 + |\gamma|,$$

and therefore the above inequality becomes

$$c/2 + |\gamma| < p/2 + c,$$

or

$$|\gamma| < \tfrac{1}{2}(p + c).$$

Since $|\gamma|$ must not exceed $c/2$ in order that the initial velocity shall be real, it follows that the inequality $z < p/2 + c$ will always be satisfied for real initial conditions.

(B). PERIODIC ORBITS ON A SURFACE OF REVOLUTION.

§ 8. *The Method of Solution.*—Let us next consider the construction of the periodic orbits described on the more general surface of revolution represented by the equation

$$F(x, y, z) = x^2 + y^2 - 2pz + 2\epsilon f(z) = 0. \tag{2}$$

The differential equations of motion have already been found, equations (3) and (5). The method of constructing the periodic solution of these equations is to first make the analytic continuation with respect to ϵ of the periodic solution for the vertical motion obtained in (A) where $\epsilon = 0$, and then substitute this solution for z in the first two equations in (3). We thus

obtain two differential equations having periodic coefficients somewhat similar to (22).

§ 9. *The Equation of Variation.*—Let us substitute in the last equation of (3)

$$\left.\begin{array}{c} z=\bar{z}+\xi,\ t-t_0=\tfrac{1}{2}\sqrt{k/g(1+\delta)}\ \tau, \\ \delta=-\tfrac{1}{2}\mu^2+\cdots, \end{array}\right\} \tag{66}$$

where $\bar{z}$ denotes the solution obtained in (21), and ξ is a function of τ which vanishes with ϵ. We obtain

$$\ddot{\xi}+\Theta_1\dot{\xi}+\Theta_2\xi=Z_0+\epsilon Z_1+\cdots+\epsilon^j Z_j+\cdots, \tag{67}$$

where the undefined terms have the following properties:

(1). The functions Θ_1 and Θ_2 are periodic functions of τ, but we shall show that it is not necessary to know the explicit values of these functions in order to solve (67).

(2). The functions $Z_0, Z_1, \cdots, Z_j$ denote power series in ξ having coefficients which are power series in μ with sums of cosines of multiples of τ in their coefficients. These functions also contain additional terms in $\dot{\xi}$ and $\dot{\xi}^2$, the former being multiplied by power series in μ with sums of sines of multiples of τ in the coefficients, the latter by similar series except that they contain cosines. The function Z_0 contains no linear terms in ξ or $\dot{\xi}$.

If we neglect the right member of (67) we obtain

$$\ddot{\xi}+\Theta_1\dot{\xi}+\Theta_2\xi=0, \tag{68}$$

which is called *the equation of variation.* The generating solution is $z=\bar{z}$, or the expression for z in (21).

Now it has been shown by Poincaré * that if the generating solution contains an arbitrary constant which does not occur in the original differential equations of motion, viz., equations (3), then a solution of the equations of variation can be obtained by differentiating the generating solution with respect to this constant. Three constants occur in (21), viz., c, μ, and t_0, the latter entering implicitly through τ. The constant c occurs in 2λ, equation (5), and therefore enters the differential equations (3). The two remaining constants, μ and t_0, do not occur in (3) and therefore may be used in applying Poincaré's theorem. Each constant yields a solution of (68) and since the differential equation is only of the second order, both its solutions may be obtained from the generating solution. Hence it is not necessary to know the coefficients Θ_1 and Θ_2 in (68) in order to solve the differential equation.

* *Les Méthodes Nouvelles de la Mécanique Celeste,* Vol. I, chap. IV. *Loc. cit.*

Consider, first, the constant t_0. Then one solution of (68) is

$$\xi = \partial \bar{z}/\partial t_0 = -2\sqrt{\frac{g}{k(1+\delta)}}\,\partial \bar{z}/\partial t = 2\sqrt{\frac{g}{k(1+\delta)}}\,\mu S(\tau), \tag{69}$$

where

$$S(\tau) = \sin \tau - \mu \sin 2\tau - \tfrac{7}{16}\mu^2 (\sin \tau - 3 \sin 3\tau) + \cdots . \tag{70}$$

Since this solution is later multiplied by an arbitrary constant, see equation (76), we may drop the constant factor of $S(\tau)$ in (69) and take

$$\xi = S(\tau) \tag{71}$$

as the solution.

Considering the constant μ, we obtain as the second solution of (68)

$$\xi = \frac{\partial \bar{z}}{\partial \mu} = \left(\frac{\partial \bar{z}}{\partial \mu}\right) + \frac{\partial \bar{z}}{\partial \tau}\,\frac{\partial \tau}{\partial \delta}\,\frac{\partial \delta}{\partial \mu}, \tag{72}$$

where the parentheses ( ) denote differentiation in so far as μ occurs explicitly in $\bar{z}$. Now

$$\left.\begin{aligned} \frac{\partial \bar{z}}{\partial \mu} &= kC(\tau) = k[\cos \tau + \mu(1 - \cos 2\tau) \\ &\qquad\qquad - \frac{21\mu^2}{16}(\cos \tau - \cos 3\tau) + \cdots], \\ \frac{\partial \tau}{\partial \delta} &= -\tfrac{1}{2}\,\frac{\tau}{1+\delta},\quad \frac{\partial \mu}{\partial \delta} = \mu[-1 + \text{power series in } \mu^2]. \end{aligned}\right\} \tag{73}$$

Therefore the solution (72) becomes

$$\xi = k[C(\tau) + \tau K S(\tau)],$$

where

$$K = \mu^2[-1 + \text{power series in } \mu^2].$$

As in the previous solution we may drop the constant factor k and take

$$\xi = C(\tau) + \tau K S(\tau), \tag{74}$$

as the second solution of (68).

The two solutions (71) and (74) constitute a fundamental set, since the determinant of these two solutions together with their derivatives is different from zero for $|\mu|$ sufficiently small, being

$$\Delta = -1 + \text{power series in } \mu. \tag{75}$$

Hence the general solution of (68) is

$$\xi = N_1 S(\tau) + N_2[C(\tau) + \tau K S(\tau)], \tag{76}$$

where N_1 and N_2 are arbitrary constants.

From the way in which $S(\tau)$ and $C(\tau)$ were derived, it is readily seen that the coefficients of $\sin(j+1)\tau$ and $\cos(j+1)\tau$ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal.

§ 10. *Construction of the Solution for the Vertical Motion.*

Let

$$\xi = \sum_{j=1}^{\infty} \xi_j \epsilon^j, \tag{77}$$

where the ξ_j are to be periodic with the period 2π in τ.

Since the initial time was chosen for $\epsilon = 0$ so that $\dot{z}(0) = 0$, it may now be chosen so that $\dot{z}(0) = 0$ when $\epsilon \neq 0$. Now $z = \bar{z} + \xi$, and as $\dot{\bar{z}}(0) = 0$, it follows that $\dot{\xi}(0) = 0$. Hence

$$\dot{\xi}_j(0) = 0, \quad j = 1, \cdots \infty. \tag{78}$$

Let (77) be substituted in (67) and let the resulting equation be denoted by (67′). It is an identity in ϵ, and on equating the coefficients of the same powers of ϵ we obtain sets of differential equations which can be integrated and the various constants of integration can be chosen, as we shall show, so as to satisfy the periodicity and initial conditions.

Two types of series occur in this integration and they are similar to $S(\tau)$ and $C(\tau)$ in (71) and (73), respectively. These series will be denoted by

$$S_j(\tau),\ S^{(j)}(\tau),\ \bar{S}_j(\tau),\ \bar{S}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

or

$$C_j(\tau),\ C^{(j)}(\tau),\ \bar{C}_j(\tau),\ \bar{C}^{(j)}(\tau),\ j = 1, 2, \cdots,$$

according as they are similar to $S(\tau)$ or $C(\tau)$, respectively.

Coefficients of ϵ. When the coefficients of ϵ to the first power are equated in (67′) we obtain

$$\ddot{\xi}_1 + \Theta_1 \dot{\xi}_1 + \Theta_2 \xi_1 = C^{(1)}(\tau). \tag{79}$$

The complementary function of this equation is the same as the solution of the equation of variation, viz.,

$$\xi_1 = n_1^{(1)} S(\tau) + n_2^{(1)} [C(\tau) + \tau K S(\tau)], \tag{80}$$

where $n_1^{(1)}$ and $n_2^{(1)}$ are the arbitrary constants. By employing the method of the variation of parameters to find the complete solution, we have

$$\left.\begin{aligned} &\dot{n}_1^{(1)} S(\tau) + \dot{n}_2^{(1)} [C(\tau) + \tau K S(\tau)] = 0, \\ &\dot{n}_1^{(1)} \dot{S}(\tau) + \dot{n}_2^{(1)} [\dot{C}(\tau) + K\{\tau \dot{S}(\tau) + S(\tau)\}] = C^1(\tau). \end{aligned}\right\} \tag{81}$$

The determinant of the coefficients of $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$ in the above equations is the same as (75), and since it is different from zero equations (79) can be solved for $\dot{n}_1^{(1)}$ and $\dot{n}_2^{(1)}$. Thus

$$\left.\begin{aligned} \dot{n}_1^{(1)} &= -(1/\Delta)C^{(1)}(\tau)[C(\tau)+\tau K\,S(\tau)], \\ \dot{n}_2^{(1)} &= (1/\Delta)C^{(1)}(\tau)\,S(\tau). \end{aligned}\right\} \tag{82}$$

Since the coefficients of the same power of μ in $C^{(1)}(\tau)$ and $C(\tau)$ are sums of cosines of the same multiples of τ, the product $C^{(1)}(\tau)C(\tau)$ will yield, in addition to periodic terms, a constant

$$\mu p_1 = \mu[p_1^{(0)} + p_1^{(1)}\mu + \cdots + p_1^{(j)}\mu^j + \cdots],$$

where $p_1^{(j)}$ are real constants. Then the integration of (82) gives

$$\left.\begin{aligned} n_1^{(1)} &= N_1^{(1)} - [\mu\,\tau\,p_1 + S^{(1)}(\tau) + \tau K\,\bar{C}^{(1)}(\tau)], \\ n_2^{(1)} &= N_2^{(1)} + \bar{C}^{(1)}(\tau), \end{aligned}\right\} \tag{83}$$

where $N_1^{(1)}$ and $N_2^{(1)}$ are the constants of integration. Since the coefficients of the sines and cosines of the highest multiples of τ in the coefficients of μ^j in $S(\tau)$ and $C(\tau)$, respectively, are equal, then it follows that $S^{(1)}(\tau)$ and $\bar{C}^{(1)}(\tau)$ have the same property. When (83) is substituted in (80) the complete solution of (79) then becomes

$$\xi_1 = N_1^{(1)}S(\tau) + N_2^{(1)}[C(\tau) + \tau KS(\tau)] - \mu\tau p_1\,S(\tau) + (1/\mu)\bar{C}_1(\tau),$$

where $\bar{C}_1(\tau)$ contains no terms independent of μ. In order that ξ_1 shall be periodic $N_2^{(1)}$ must be given the value

$$N_2^{(1)} = \frac{\mu p_1}{K} = \frac{1}{\mu}\ (\text{power series in } \mu), \tag{84}$$

and from the initial conditions (78) it follows that

$$N_1^{(1)} = 0.$$

Hence

$$\xi_1 = (1/\mu)C_1(\tau), \tag{85}$$

where $C_1(\tau)$, like $\bar{C}_1(\tau)$, contains no terms independent of μ.

Coefficients of ϵ^2. Equating the coefficients of ϵ^2 in (67) gives the differential equation

$$\ddot{\xi}_2 + \Theta_1\dot{\xi}_2 + \Theta_2\xi_2 = (1/\mu)C^{(2)}(\tau). \tag{86}$$

Except for the factor $1/\mu$ in the right member, this equation is similar to (79). The general solution is obtained in the same way as at the preceding step and is found to be

$$\xi_2 = N_1^{(2)} S(\tau) + N_2^{(2)} [C(\tau) + \tau K S(\tau)] + \tau p_2 S(\tau) + (1/\mu) \bar{C}^{(2)}(\tau),$$

where $N_1^{(2)}$ and $N_2^{(2)}$ are the constants of integration, and p_2 is a power series in μ. In order to satisfy the periodicity and initial conditions we must put

$$N_2^{(2)} = \frac{p_2}{K} = \frac{1}{\mu^2} \text{ (power series in } \mu), \; N_1^{(2)} = 0.$$

Hence

$$\xi_2 = (1/\mu^2) C_2(\tau),$$

where $C_2(\tau)$, contains no terms independent of μ.

The remaining steps of the integration can be carried on in the same way and by an induction to the general term it can be shown that

$$\xi_n = (1/\mu^n) C_n(\tau),$$

where $C_n(\tau)$ as at the previous steps, contains no terms independent of μ.

On substituting the solutions for the various ξ_j in (77) we obtain

$$\xi = \sum_{j=1}^{\infty} C_j(\tau)(\epsilon/\mu)^j. \tag{87}$$

By applying Macmillan's theorem, quoted in § 3, it is found that the solution (87) converges for $|\epsilon|$ sufficiently small.

When (87) is substituted in (66), the solution for the vertical motion becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau)(\epsilon/\mu)^j. \tag{88}$$

Since ϵ and μ are both arbitrary we may put $\epsilon = \rho\mu$ and therefore (88) becomes

$$z = \bar{z} + \sum_{j=1}^{\infty} C_j(\tau)\rho^j. \tag{89}$$

This solution converges for $|\mu|$ and $|\rho|$ sufficiently small.

§ 10. *The Horizontal Motion.* When the first two equations of (3) are transformed by the substitution

$$t - t_0 = \tfrac{1}{2}\sqrt{(k/g)(1+\delta)}\;\tau, \quad \delta = -\tfrac{1}{2}\mu^2 + \cdots,$$

already used in (A), and the value of z obtained in (89) is substituted in 2λ, these differential equations become

$$\left.\begin{aligned} \ddot{x} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots]\,x = 0, \\ \ddot{y} + [\phi_0 + \phi_1\rho + \phi_2\rho^2 + \cdots]\,y = 0, \end{aligned}\right\} \tag{90}$$

where each ϕ_j is a power series in μ with sums of cosines of multiples of τ in the coefficients. The function ϕ_0 has the same value as the coefficient of x in (22).

Now let $\rho = \sigma\mu$. Then the coefficients in (90) can be rearranged as power series in μ and the differential equations take the same form as (22). Hence the solutions are

$$x = L_1 e^{i\beta\tau} v_1 + L_2 e^{-i\beta\tau} v_2, \quad y = M_1 e^{i\beta\tau} v_1 + M_2 e^{-i\beta\tau} v_2, \tag{91}$$

where L_1, L_2, M_1, and M_2 are the constants of integration, and β, v_1, and v_2 are similar in form to α, u_1, and u_2, respectively, of (39), (52), or (60) according to the value of $\sqrt{k/p}$.

When the initial values $\dot{x}(0) = y(0) = 0$ are imposed on (91), it follows that

$$[i\beta + \dot{v}_1(0)]\, [L_1 - L_2] = 0, \quad M_1 + M_2 = 0.$$

Since $i\beta + \dot{v}_1(0) \neq 0$ for $|\, \mu \,|$ sufficiently small, then

$$L_1 = L_2 = L, \text{ say,}$$

and, from the second equation,

$$M_1 = - M_2 = M, \text{ say.}$$

Hence the solutions (91) become

$$x = L[e^{i\beta\tau} v_1 + e^{-i\beta\tau} v_2], \quad y = M[e^{i\beta\tau} v_1 - e^{-i\beta\tau} v_2].$$

The constants L and M can be determined as in § 7, and it is found that L is real while M is purely imaginary. Suppose $M = iN$. Then

$$x = 2L[\cos\beta\tau\{1 + \sum_{j=1}^{\infty} \sum_{l=0}^{j} \mu^j a_l^{(j)} \cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty} \sum_{l=1}^{j} \mu^j b_l^{(j)} \sin l\tau\}],$$

$$y = 2N[\sin\beta\tau\{1 + \sum_{j=1}^{\infty} \sum_{l=0}^{j} \mu^j a_l^{(j)} \cos l\tau\} - \sin\beta\tau\{\sum_{j=1}^{\infty} \sum_{l=1}^{j} \mu^j b_l^{(j)} \sin l\tau\}],$$

where $a_l^{(j)}$ and $b_l^{(j)}$ are real constants. These equations together with equation (89) are the solutions of the equations of motion of the problem under consideration.

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